1961

Characterizations of norms given by inner products

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CHARACTERIZATIONS OF NORMS GIVEN BY INNER PRODUCTS

by
James P. Crawford

A Thesis

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Master of Science

Lehigh University

1961

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 24/61
(date)

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Acknowledgement

The author wishes to acknowledge the patient advice and encouragement given him by his advisor, Dr. Albert Wilansky, during the writing of this thesis.

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INTRODUCTION

We are concerned with linear spaces over real or complex fields of scalars. We will generally designate the elements of the space with Roman letters in lower case and the scalars with small Greek letters. Exceptions will be noted.

A norm, $||.||$, in a (real or complex) linear space A is a function whose domain is the set of elements of the space A and whose range is the set of non-negative real numbers. A norm has the following addition properties:

N_1. $||a|| = 0$ if and only if $a = 0$.

N_2. $||\alpha a|| = |\alpha| \, ||a||$ for all scalars α.

N_3. $||a+b|| \leq ||a|| + ||b||$.

It is easily seen that a norm on a linear space will perform as a metric simply by letting $d(a,b) = ||a-b||$.

An inner-product, $(.,.)$ on a linear space A is a function whose domain is $A \times A$ and whose range is the field of scalars of A. It has the following additional properties:

I_1. $(a,b) = \overline{(b,a)}$.

I_2. $(a+c,b) = (a,b) + (c,b)$.

I_3. $(\alpha a,b) = \alpha(a,b)$ for all scalars α.

I_4. $(a,a) \geq 0$ for all $a \in A$. It is real because of I_1.

Equality holds if and only if $a = 0$.

It is noteworthy that if A is a real linear space the Hermitian symmetry of I_1 becomes simple symmetry.

Before establishing the important connection between inner-products and norms, we consider a property of inner-products called the Cauchy-Schwarz inequality. The Cauchy-Schwarz Inequality:

$$|(a,b)|^2 \leq (a,a) \cdot (b,b)$$

The equality holds only if a and b are linearly dependent. Because of property I_4 and because properties I_1 and I_3 imply $(\alpha a, \alpha a) = \alpha\bar{\alpha}(a,a)$, the function $\sqrt{(a,a)}$ is a candidate for a norm on the inner product space.

A complete normed linear space is called a Banach space. A normed linear space whose norm is given by an inner product is called an inner-product space. A Hilbert space is a complete inner-product space.

The problem we wish to investigate can now be stated. Given a normed linear space A, is it possible to define on A an inner product such that $||a|| = \sqrt{(a,a)}$?

It is known that it is always possible to define an inner product on any linear space by making use of the Hamel basis. It is also known that it is not always possible to produce an inner product which gives the norm. The ability to produce such an inner product depends on various properties of the norm, and it is these properties of the norm which we wish to investigate.

Since we are trying to characterize inner-product spaces among the larger class of normed linear spaces, we will give conditions both necessary and sufficient for the existence of an inner-product giving the norm in a normed linear space.

Chapter 1

The Elliptic Unit Sphere and the Parallelogram Law

We begin with a rather simple geometrical characterization of a norm given by an inner product in a normed real linear space.

Theorem 1.1: Let A be a two dimensional normed real linear space and let a, b be a basis for A. Then A is an inner product space if and only if $S = \{(\alpha,\beta) \mid ||\alpha a+\beta b|| = 1\}$ is an ellipse.

Proof: Suppose A is a real two dimensional inner product space. If α and β are scalars such that $(\alpha,\beta)\in S$ then $1 = ||\alpha a+\beta b|| = (\alpha a+\beta b,\alpha a+\beta b) = \alpha^2(a,a) + 2\alpha\beta(a,b) + \beta^2(b,b)$ $= \alpha^2||a||^2 + 2\alpha\beta(a,b) + \beta^2||b||^2$. Since a and b are linearly independent, the Cauchy-Schwartz inequality gives us

$$(a,b)^2 - ||a||^2\ ||b||^2 < 0.$$

Thus the equation $||a||^2\alpha^2 + 2(a,b)\alpha\beta + ||b||^2\beta^2 = 1$ is the equation of an ellipse.

Now suppose A is a two dimensional normed real linear space with a and b linearly independent vectors in A. Suppose further that $S = \{(\alpha,\beta) \mid ||\alpha a+\beta b|| = 1\}$ is an ellipse whose equation is $x\alpha^2 + 2y\alpha\beta + z\beta^2 = 1$. For any vectors c and d in A we have $c = \alpha_1 a + \beta_1 b$, $d = \alpha_2 a + \beta_2 b$.

Define $(c,d) = x\alpha_1\alpha_2 + y\ (\alpha_1\alpha_2 + \beta_1\beta_2) + z\beta_1\beta_2$. We shall show that (c,d) is an inner product on A and that $(c,c) = ||c||^2$:

I_1. That $(c,d) = (d,c)$ is clear.

I_2. Suppose $e = \alpha_3 a + \beta_3 b$. Then $(c+e,d)$

$$= x(\alpha_1+\alpha_3)\alpha_2 + y[(\alpha_1+\alpha_3)\beta_2 + (\beta_1+\beta_3)\alpha_2]$$
$$+ z(\beta_1+\beta_3)\beta_2 = x\alpha_1\alpha_2 + y(\alpha_1\beta_2+B_1\alpha_2)$$
$$+ z\beta_1\beta_2 + x\alpha_3\alpha_2 + y(\alpha_3\beta_2+\beta_3\alpha_2) + z\beta_3 B_2$$
$$= (c,d) + (e,d).$$

I_3. That $(\gamma c,d) = \gamma(c,d)$ for all scalars γ is clear.

I_4. If c is any element of A, $c = \alpha a + \beta b \neq 0$, then

$$\left(\frac{c}{||c||}, \frac{c}{||c||}\right) = \frac{x\alpha^2}{||c||^2} + \frac{2y\alpha\beta}{||c||^2} + \frac{z\beta^2}{||c||^2} = 1.$$

Thus $(c,c) = ||c||^2 \geq 0$. $(c,c) = 0$ only if $c = 0$. This concludes the proof that (c,d) is an inner product giving the norm in A.

We proceed now to the parallelogram law, a characterization basic to our work, which was first proved by P. Jordan and J. von Neumann [15].

Jordan and von Neumann actually proved this condition was necessary and sufficient for an invariant metric space to be an inner product space. An invariant metric differs from a norm only in that the property N_2 is replaced by $||\alpha a|| \to 0$ as $\alpha \to 0$ and $||ia|| = ||a||$. We do not need a result this strong.

To motivate the theorem, we let A be a real inner product space and let (a,b) denote the inner product of a and b. Then $||a|| = \sqrt{(a,a)}$

(*) $||a+b||^2 = (a+b,a+b) = (a,a) + (a,b) + (b,a) + (b,b)$
$= ||a||^2 + 2(a,b) + ||b||^2$

(**) $||a-b||^2 = (a-b,a-b) = (a,a) + (a,-b) + (-b,a) + (-b,-b)$
$= ||a||^2 - 2(a,b) + ||b||^2$

Adding (*) and (**) we get

$$||a+b||^2 + ||a-b||^2 = 2\{||a||^2 + ||b||^2\} \text{ for all } a,b\epsilon A$$

This equation has the following geometrical interpretation: The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides of the parallelogram. The presence of this "parallelogram law" is any real inner product space indicates the necessity of the condition in the following theorem.

Theorem 1.2: ([15], Theorem 1) Let A be a normed real linear space, then A is an inner product space if and only if for each $a,b\epsilon A$

$$||a+b||^2 + ||a-b||^2 = 2||a||^2 + 2||b||^2$$

Proof: The necessity of the condition is proved above. To prove sufficiency, assume that A is a normed real linear space and that $||a+b||^2 + ||a-b||^2 = 2\{||a||^2 + ||b||^2\}$ for all a, $b\epsilon A$. By subtracting (**) from (8) above we see that, if A holds any hope of being an inner product space, the inner product must satisfy the relation

$$4(a,b) = ||a+b||^2 - ||a-b||^2$$

We therefore anticipate our result by writing $(a,b) = \frac{1}{4}\{||a+b||^2 - ||a-b||^2\}$ and then we will justify our notation by showing that (a,b) is indeed a real inner-product.

(1) Note first that $(a,b) = (b,a)$, which is simply I_1 for a real inner product.

(2) Clearly $(a,a) = ||a||^2$ which shows that I_4 is satisfied.

(3) Note that $(\alpha a, \alpha b) = \dfrac{||\alpha a+\alpha b||^2 - ||\alpha a-\alpha b||^2}{4} = \alpha^2(a,b)$ where α is any real scalar.

$$(a,b) + (c,b) = \frac{||a+b||^2 - ||a-b||^2 + ||c+b||^2 - ||c-b||^2}{4}$$

Using the parallelogram law with $a = a+b$ and $b = c+b$ and again with $a = a-b$ and $b = c-b$ we get

$$= \frac{||a+b||^2 + ||c+b||^2}{4} - \frac{||a-b||^2 + ||c-b||^2}{4}$$

$$= \frac{||a+b+c+b||^2 + ||a+b-c-b||^2}{8} - \frac{||a-b+c-b||^2 + ||a-b-c+b||^2}{8}$$

$$= \frac{||a+c+2b||^2 - ||a+c-2b||^2}{8} = \frac{1}{2}\left\{\left|\left|\frac{a+c}{2} + b\right|\right|^2 - \left|\left|\frac{a+c}{2} - b\right|\right|^2\right\}$$

$$= 2\left(\frac{a+c}{2}, b\right) = \frac{1}{2}(a+c, 2b)$$

Take $c = 0$, then since $(c,b) = 0$ we have

(4) $(a,b) = 2(\frac{a}{2}, b) = \frac{1}{2}(a, 2b)$. In (4) replace a by $a+c$ to get $(a+c,b) = \frac{1}{2}(a+c,2b) = (a,b) + (c,b)$. Thus (a,b) is additive in a, which establishes I_2.

(5) To establish I_3, let $S = \{\alpha \mid (\alpha a,b) = \alpha(a,b)\}$. Clearly 0 and $1 \in S$ and $n \in S$, n a positive integer, implies that

$n+1 \in S$ because of the additivity of (a,b) in the first variable. By induction then, all positive integers belong to S.

Suppose n is a positive integer. Then $(na,b) + (-na,b) = (na-na, b) = (0,b) = 0$. Thus $(-na, b) = -(na, b) = -n(a,b)$. Therefore all integers belong to S.

Now let p be an integer, $p \neq 0$. Then $p(\frac{1}{p} a,b) = (\frac{p}{p} a,b) = (a,b)$. Thus $(\frac{1}{p} a,b) = \frac{1}{p}(a,b)$. S therefore contains all of the rational numbers.

Since the rationals are dense in the reals, it remains only to show that $(\alpha a,b)$ is continuous in α for fixed a and b, and this will establish that S indeed consists of all real numbers.

Note that, by N_3, $||a-b|| \geq |\ ||a|| - ||b||\ |$. Therefore $|\ ||\alpha a \pm b|| - ||\beta a \pm b||\ | \leq ||(\alpha-\beta)a|| = |\alpha-\beta|\ ||a||$. Thus $\lim_{\alpha \to \beta} ||\alpha a \pm b|| = ||\beta a \pm b||$. Now let β be any real number, let $\{\alpha_n\}$ be a sequence of rational numbers converging to β, then $(\beta a,b) = \lim_{n \to \infty} (\alpha_n a,b) = \lim_{n \to \infty} \alpha_n(a,b) = \beta(a,b)$. Thus $\beta \in S$ and S must therefore consist of all the real numbers. This establishes I_3 and completes the proof that (a,b) is an inner product. That the inner product gives the norm is shown in (2). Thus A is a real inner product space.

We turn now to a theorem which we will use often to show that some characteristics of the norm which are necessary and sufficient conditions that a normed real linear space be a real inner-product space are also necessary and sufficient that a normed complex linear space be a complex inner product space.

If A is a complex linear space, then there is associated with A a real linear space B whose elements are the elements of A and whose addition is the addition of A. If α is a real number, the multiple αa, $a \in B$ is equal to t e multiple $(\alpha+i\cdot 0)a$ in A. If A is normed, the same norm is employed for B. We shall call B the real space associated with A.

<u>Theorem</u> 1.3.([5], Theorem 7.2) <u>Let</u> A <u>be</u> <u>a</u> <u>normed</u> <u>complex</u> <u>linear</u> <u>space</u> <u>and</u> <u>let</u> B <u>be</u> <u>the</u> <u>associated</u> <u>normed</u> <u>real</u> <u>linear</u> <u>space</u>. <u>Then</u> A <u>is</u> <u>a</u> <u>complex</u> <u>inner</u> <u>product</u> <u>space</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> B <u>is</u> <u>a</u> <u>real</u> <u>inner</u> <u>product</u> <u>space</u>.

<u>Proof</u>. If A is a complex inner product space, let $[a,b]$ indicate the inner product in A. Let B be the associated normed real linear space and let $(a,b) = \mathrm{Re}([a,b])$, the real part of $[a,b]$.

1. $(b,a) = \mathrm{Re}([a,b]) = \mathrm{Re}(\overline{[a,b]}) = \mathrm{Re}([a,b]) = (a,b)$.
2. $(a+c,b) = \mathrm{Re}([a+c,b]) = \mathrm{Re}([a,b] + [c,b]) = (a,b) + (c,b)$.
3. $(\alpha a,b) = \mathrm{Re}([\alpha a,b]) = \mathrm{Re}(\alpha[a,b]) = \alpha\mathrm{Re}([a,b]) = \alpha(a,b)$ for all real scalars α.

4. $(a,a) = \mathrm{Re}([a,a]) = \mathrm{Re}(||a||^2) = ||a||^2$. Thus (a,b) is a real inner product and B is a real inner space.

Now let A be a normed complex linear space whose associated real space B is a real inner product space with inner product (a,b). Since we wish to define a complex inner product, $[a,b]$, (which is necessarily real-linear in a) an A whose real part is (a,b), we have only one choice of definition; namely, let $[a,b] = (a,b) - i(ia,b)$. We now show that $[a,b]$ is an inner product.

Recall that, since B is a real inner product space and the norm in B is the same as the norm in A,

$$(a,b) = \frac{||a+b||^2 - ||a-b||^2}{4}$$

and

$$(ia,ib) = \frac{||ia+ib||^2 - ||ia-ib||^2}{4} = \frac{||a+b||^2 - ||a-b||^2}{4}$$

$$= (a,b).$$

(1) $[b,a] = (b,a) - i(ib,a) = (a,b) - i(i\cdot ib,ia) =$

$$= (a,b) - i(-b,ia) = (a,b) + i(b,ia) =$$

$$= \overline{[a,b]}.$$

(2) $[a+c,b] = (a+c,b) - i(ia+ic,b) = (a,b) - i(ia,b) + (c,b) - i(ic,b) = [a,b] + [c,b]$.

(3) If z is a complex number, $z = \alpha + \beta i$, α,β real, then $[za,b] = (\alpha a+\beta ia,b) - i(\alpha ia-\beta a,b) = \alpha(a,b) +$

$\beta(ia,b) - i\alpha(ia,b) + i\beta(a,b) = (\alpha+i\beta)(a,b) - (\alpha+i\beta)i(ia,b) = z[a,b]$.

(4) $[a,a] = (a,a) - i(ia,a)$. But $(ia,a) = (-a,ia) = -(ia,a)$ thus $(ia,a) = 0$. Therefore $[a,a] = (a,a) = ||a||^2$. A is therefore a complex inner product space.

<u>Theorem</u> 1.4. <u>Let</u> A <u>be</u> <u>a</u> <u>normed</u> <u>complex</u> <u>linear</u> <u>space</u>. <u>Then</u> A <u>is</u> <u>an</u> <u>inner</u> <u>product</u> <u>space</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $||a+b||^2 + ||a-b||^2 = 2||a||^2 + 2||b||^2$ <u>for</u> <u>all</u> $a,b \in A$.

<u>Proof</u>. Let A be a complex inner product space. By simply expanding the inner products giving the norms and adding as in Theorem 1, the parallelogram law results.

Now let A be a normed complex linear space in which the parallelogram law holds for all $a,b \in A$. Let B be the real linear space associated with A. Then, since the norm is the same in both spaces, $||a+b||^2 + ||a-b||^2 = 2\{||a||^2 + ||b||^2\}$ for all $a,b \in B$. Thus be Theorem 1.2 B is a real inner product space and by Theorem 1.3 A is a complex inner product space.

<u>Corollary</u> 1.4.1. <u>Let</u> A <u>be</u> <u>a</u> (real or complex) <u>normed</u> <u>linear</u> <u>space</u> <u>of</u> <u>dimension</u> <u>greater</u> <u>than</u> <u>or</u> <u>equal</u> <u>to</u> <u>two</u>. <u>Then</u> A <u>is</u> <u>an</u> <u>inner</u> <u>product</u> <u>space</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> <u>every</u> <u>two</u> <u>dimensional</u> <u>subspace</u> <u>of</u> <u>A</u> <u>is</u> <u>an</u> <u>inner</u> <u>product</u> <u>space</u>.

Proof: Let A be an inner product space and let B be a two dimensional subspace of A. The inner-product in A is an inner-product in B giving the norm, thus B is an inner-product space.

Now suppose A is a normed linear space of dimension greater than or equal to two in which every two dimensional subspace is an inner-product space. Let $a, b \in A$. Then a, b belong to some two dimensional linear subspace B; and, since B is an inner-product space, $||a+b||^2 + ||a-b||^2 = 2\,\{||a||^2 + ||b||^2\}$. Since the choice of a and b from A was arbitrary, the parallelogram law is satisfied in A. Therefore A is an inner-product space.

Because of Theorem 1.2 and 1.4 this proof works as well in either real or complex linear spaces.

The following theorem by Ficken depends directly upon Theorem 1.2 and shows a familiar geometrical property which characterizes inner-product spaces among normed linear spaces. The geometrical property referred to is as follows: "If two points are symmetrically located with respect to a line through the origin, then these points are equidistant from the origin."

Theorem 1.5 ([9], Theorem 1): Let A be a normed real linear space. Then A is an inner-product space if and only if $||a|| = ||b||$ implies $||\alpha a+\beta b|| = ||\beta a+\alpha b||$ for all real scalars α, and β and elements a and b of A.

Proof: Suppose A is a real inner-product space. Then, if $||a|| = ||b||$, $a,b \in A$; $||\alpha a+\beta b||^2 = (\alpha a+\beta b, \alpha a+\beta b)$ $= \alpha^2||a||^2 + 2\alpha\beta(a,b) + \beta^2\ ||b||^2 = \alpha^2||b||^2$ $+ 2\alpha\beta(b,a) + \beta^2\ ||a||^2 = (\beta a+\alpha b, \beta a+\alpha b) = ||\beta a+\alpha b||^2$ for any real scalars α and β. Therefore the condition of the theorem is necessary.

To prove the sufficiency of the condition, we require the following lemmas to show that the condition of the theorem implies that the parallelogram law holds. In each of the lemmas the condition of the theorem will be tacitly an additional hypothesis and the vectors discussed will be elements of A.

Lemma 1: If $||a|| = ||b|| = ||\frac{a+b}{2}||$, then $a = b$.

Proof: $||a|| = ||b|| = ||\frac{a+b}{2}||$ implies $||a+b|| = ||2b||$ and, using the assumption of the theorem, it follows that

$$||a+b - 2(2b)|| = ||2(a+b) - 2b|| = ||2a||$$

or $$||a-b-2b|| = 2||a|| = 2||b||.$$

Let $P(n)$ be the proposition that $||(2n+1)(a-b)-2b||$ $= 2||a|| = 2||b||$. We have just shown that $p(0)$ is true. Proceeding with the induction, assume that for some $n \geq 0$, $p(n)$ is true. Using the condition of the theorem again

$$||2\{(2n+1)(a-b)-2b\} + (2n+3)2b||$$

$$= ||(2n+3)\{(2n+1)(a-b)-2b\} + 2(2b)||$$

or $$||2(2n+1)a|| = ||(2n+3)\{(2n+1)a-(2n+3)b\} + 4b||$$

$$= ||(2n+3)(2n+1)a-(2n+3)^2b + 4b||$$

$$= (2n+1)||(2n+3)(a-b)-2b||.$$

Thus $2||a|| = ||(2n+3)(a-b)-2b||$. By induction then $||(2n+1)(a-b)-2b|| = 2||a|| = 2||b||$ for all non-negative integers n.

By the triangle inequality (N_3)

$$2||a|| \geq (2n+1)\,||a-b|| - 2||b||$$

or $\quad ||a-b|| \leq \dfrac{2(||a|| + ||b||)}{2n+1} = \dfrac{4||b||}{2n+1}$ for all $n = 0,1,2,\ldots$

Thus $||a-b|| = 0$ and $a = b$.

Lemma 2: $||a||b \neq ||b||a$ implies $||a+b|| < ||a|| + ||b||$.

Proof: $||a+b|| \leq ||a|| + ||b||$ is always true (N_3).

Let $a, b \in A$ and suppose $||a|| \leq ||b||$. Then $||a+b||$

$$= \left|\left| a + ||a|| \frac{b}{||b||} + (||b|| - ||a||)\frac{b}{||b||} \right|\right|$$

$$\leq \left|\left| a + ||a|| \frac{b}{||b||} \right|\right| + \Big|\, ||b|| - ||a|| \,\Big|$$

Since $||b||a \neq ||a||b$ implies $a \neq ||a|| \frac{b}{||b||}$, Lemma 1

says $\left|\left| a + ||a|| \frac{b}{||b||} \right|\right| < 2||a||$.

Thus $||a+b|| < 2||a|| + ||b|| - ||a|| = ||a|| + ||b||$.

Lemma 3: For any $f, g \in A$ and positive real number β there are not more than two distinct real values of α such that

$$||f + \alpha(g-f)|| = \beta.$$

Proof: If f and g are linearly dependent, there are clearly at most two such values of α. Assume therefore that f and g are linearly independent. If the assertion of the lemma is false, we can find vectors

$$a = f + \alpha_1(g-f)$$
$$b = f + \alpha_2(g-f)$$
$$c = f + \alpha_3(g-f)$$

such that $||a|| = ||b|| = ||c|| = \beta$. Clearly, since these vectors are linear combinations of the vectors f and g, the set a,b,c is not a linearly independent set. But there are at least two linearly independent elements in the set $\{a,b,c\}$. [Suppose this were not the case. Then $a = \gamma_1 b = \gamma_2 c$ for some scalars γ_1, γ_2. Since $||a|| = ||b|| = ||c||$, $\gamma_1 = \pm 1$ and $\gamma_2 = \pm 1$. If $\gamma_1 = 1$, then $\alpha_1 = \alpha_2$. If $\gamma_2 = 1$, then $\alpha_1 = \alpha_3$. If $\gamma_1 = \gamma_2 = -1$, then $\alpha_1 = -\alpha_2 = -\alpha_3$; that is, $\alpha_2 = \alpha_3$. Since the α_i's were assumed distinct this gives a contradiction.] For definiteness suppose that a and b are linearly independent. Then $c = \gamma_1 a + \gamma_2 b$ for some real scalars γ_1 and γ_2. Then $(1-\alpha_3)f + \alpha_3 g = \gamma_1(1-\alpha_1)f + \gamma_1\alpha_1 g + \gamma_2(1-\alpha_2)f + \gamma_2\alpha_2 g$. Equating coefficients we get $1 - \alpha_3 = \gamma_1(1-\alpha_1) + \gamma_2(1-\alpha_2) = \gamma_1 - \gamma_1\alpha_1 + \gamma_2 - \gamma_2\alpha_2$ and $\alpha_3 = \gamma_1\alpha_1 + \gamma_2\alpha_2$. ~~Adding we get $1 = \gamma_1 + \gamma_2$.~~

Suppose that γ_2 is negative. (If $\gamma_2 = 0$, then $c = a$ which is impossible.) Using the result of Lemma 2, we observe that $||a-b||b \neq ||b||(a-b)$ implies $||a-b+b|| = ||a|| < ||a-b|| + ||b||$ or $||a-b|| > ||a|| - ||b||$. Thus, since a and b are linearly independent, $\beta = ||c|| = ||\gamma_1 a+\gamma_2 b|| > \gamma_1||a|| - |\gamma_2|\cdot||b|| = (\gamma_1+\gamma_2)\beta = \beta$; and there is perfect symmetry. Since both γ_1 and γ_2 are positive, we use the result of Lemma 2 and the linear independence of a and b to get $\beta = ||c|| = ||\gamma_1 a+\gamma_2 b|| < \gamma_1||a|| + \gamma_2||b|| = (\gamma_1+\gamma_2)\beta = \beta$. This contradiction establishes the lemma.

We turn now to the proof of the theorem. We wish to show that the parallelogram law holds in a normed real linear space A in which $a,b \in A$ and $||a|| = ||b||$ imply $||\alpha a+\beta b|| = ||\beta a+\alpha b||$ for all real scalars α and β.

If a and b are linearly dependent elements of A it is very easy to show that $||a+b||^2 + ||a-b||^2 = 2||a||^2 + ||b||^2$. We will therefore confine our considerations to any two linearly independent elements a and b of A.

Using the hypothesized condition of the theorem

$$||a+b|| = ||a-b+2b|| = \left|\left| \frac{||a-b||(a-b)}{||a-b||} + \frac{2||b||2b}{2||b||} \right|\right|$$

$$= \left|\left| 2||b||\frac{(a-b)}{||a-b||} + ||a-b||\frac{b}{||b||} \right|\right|$$

$$= \left|\left| \frac{2||b||a}{||a-b||} + \frac{(||a-b||^2-2||b||^2)b}{||b||\cdot||a-b||} \right|\right|$$

$$= \left|\left| \frac{2||b||^2a + (||a-b||^2-2||b||^2)b}{||b||\cdot||a-b||} \right|\right|.$$

Thus

(1) $||b||\cdot||a+b||\cdot||a-b|| = ||2||b||^2a + (||a-b||^2-$

$2||b||^2)b||$.

Similarly

$$||a+b|| = ||b-a+2a|| = \left|\left| \; ||b-a||\frac{(b-a)}{||b-a||} + 2||a||\frac{2a}{2||a||} \right|\right|$$

$$= \left|\left|\frac{2||a||(b-a)}{||b-a||} + \frac{||b-a||a}{||a||}\right|\right|$$

$$= \left|\left| \frac{2||a||b}{||b-a||} + \left(\frac{||b-a||}{||a||} - \frac{2||a||}{||b-a||}\right)a \right|\right|$$

$$= \left|\left| 2||a||\cdot||b||\frac{b}{||b||\cdot||b-a||} + \right.$$

$$\left. (||b-a||^2-2||a||^2)\frac{a}{||a||\cdot||b-a||} \right|\right|$$

$$= \left|\left|(||b-a||^2-2||a||^2)\frac{b}{||b||\cdot||b-a||} + \right.$$

$$\left. \frac{2||a||\cdot||b||a}{||a||\cdot||b-a||} \right|\right|$$

$$= \left|\left|\frac{(||b-a||^2-2||a||^2b}{||b||\cdot||b-a||} + \frac{2||b||^2a}{||b||\cdot||b-a||} \right|\right|.$$

Thus

(2) $||b||\cdot||a+b||\cdot||a-b|| = ||2||a||\cdot||b||a+(||a-b||^2-$

$2||a||^2)b||$.

Also.

$$||a-b|| = ||a+b-2b|| = \left|\left| \, ||a+b|| \frac{(a+b)}{||a+b||} - \frac{2||b||}{2||b||} 2b \right|\right|$$

$$= \left|\left| \frac{2||b||(a+b)}{||a+b||} - \frac{||a+b||\,b}{||b||} \right|\right|$$

$$= \left|\left| \frac{2||b||^2 a + (2||b||^2 - ||a+b||^2)b}{||b||\cdot||a+b||} \right|\right| .$$

Thus

(3) $||b||\cdot||a+b||\cdot||a-b|| = ||2||b||^2 a +$
$(2||b||^2-||a+b||^2)b||.$

And finally

$$||b-a|| = ||b+a-2a|| = \left|\left| \, ||b+a|| \frac{(b+a)}{||b+a||} - 2||a|| \frac{2a}{2||a||} \right|\right|$$

$$= \left|\left| 2||a|| \frac{(b+a)}{||b+a||} - \frac{||b+a||\,a}{||a||} \right|\right|$$

$$= \left|\left| \frac{2||a||\,b}{||b+a||} + \left(\frac{2||a||}{||b+a||} - \frac{||b+a||}{||a||}\right)a \right|\right|$$

$$= \left|\left| \frac{2||a|| \; ||b||\,b}{||b||\cdot||b+a||} + \frac{(2||a||^2-||b+a||^2)a}{||a||\cdot||b+a||} \right|\right|$$

$$= \left|\left| \frac{(2||a||^2-||b+a||^2)b}{||b||\cdot||b+a||} + \frac{2||a||\cdot||b||\,a}{||a||\cdot||b+a||} \right|\right|$$

Thus

(4) $||b||.||a+b||.||a-b|| = ||2||b||^2 a + (2||a||^2-||a+b||^2)b||$

Now apply Lemma 3 with $f = 2||b||^2 a$, $f-g = b$, and $\beta = ||b||.||a+b||.||a-b||$. Looking at (1), (2), (3), and (4) we see that of the four numbers

$$\alpha_1 = ||a-b||^2 - 2||b||^2 \qquad \alpha_2 = ||a-b||^2 - 2||a||^2$$
$$\alpha_3 = 2||b||^2 - ||a+b||^2 \qquad \alpha_4 = 2||a||^2 - ||a+b||^2$$

at most two are distinct.

Case I: If $\alpha_1 = \alpha_4$ or $\alpha_2 = \alpha_3$ we get the parallelogram law immediately and the proof is complete.

Case II: If $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$, we have $||a+b||^2 + ||a-b||^2 = 4||a||^2 = 4||b||^2$ which is simply the parallelogram law again.

Case III: The only remaining possibility is that $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. This implies that $||a|| = ||b||$. Since a and b are arbitrary independent vectors in A, replace a and b in Cases I and II by the linearly independent vectors a+b and a-b, respectively. From the results of Cases I and II we see that the parallelogram law must hold for these vectors, a+b and a-b. Thus
$$||(a+b) + (a-b)||^2 + ||(a+b) - (a-b)||^2 = 2||a+b||^2 + 2||a-b||^2$$
or $4||a||^2 + 4||b||^2 = 2||a+b||^2 + 2||a-b||^2$ which is again the parallelogram law for a and b. We then take the replacements to Case III where we conclude $||a+b|| = ||a-b||$.

It now remains only to show that $||a|| = ||b||$ and $||a+b|| = ||a-b||$ imply the parallelogram law. Suppose $\alpha_2 = \alpha_1 > 0$, then using (1), the linear independence of a and b, and Lemma 2; we get

$||b|| \, ||a+b|| \, ||a-b|| < 2||b||^2||a|| + (||a-b||^2-2||b||^2)||b||$

or $||b||.||a+b.||a-b|| < 2||b||^3 + ||a-b||^2||b|| - 2||b||^3$.

Hence $||b||.||a+b||.||a-b|| < ||b|| \, ||a-b||^2$

giving $||a+b|| < ||a-b||$ contradicting our immediate hypothesis that $||a+b|| = ||a-b||$.

On the other hand if $\alpha_2 = \alpha_1 \;\; 0$, then using (1) and a variation on Lemma 2 we get $||b||.||a+b||.||a-b||$ $> 2||b||^2||a|| - (2||b||^2-||a-b||^2)||b||$ or $||b||.||a+b||.||a-b|| > 2||b||^3 - 2||b||^3 + ||a-b||^2||b||$.

Hence $||a+b|| > ||a-b||$ again a contradiction. Thus $\alpha_1 = \alpha_2 = 0$. In a similar manner working with α_3 and (3) it can be shown that $||a|| = ||b||$ and $||a+b|| = ||a-b||$ imply that $\alpha_3 = \alpha_4 = 0$.

Thus, referring to α_1, α_2, α_3 and α_4 with $\alpha_1 = \alpha_2 = \alpha \;\; = \alpha_4 = 0$, we get $||a-b||^2 = 2||b||^2$ and $||a+b||^2 = 2||a||^2$. Adding these gives the parallelogram law again completing the proof of the theorem.

Corollary 1.5.1: In order that a normed complex linear space A be an inner product space it is necessary and sufficient that $a, b \in A$ and $||a|| = ||b||$ imply $||\alpha a+\beta b|| = ||\beta a+\alpha b||$ for all real scalars α and β.

Proof: We prove sufficiency first. Let A be the normed complex linear space described in the corollary and let B be the associated real linear space described in Theorem 1.3. Since "$||a|| = ||b||$ implies $||\alpha a+\beta b|| = ||\beta a+\alpha b||$

for all real scalars α; β" is true in A, it must also be true in B. Thus by Theorem 1.5, B is an inner product space; and by Theorem 1.3 so is A.

Now suppose that A is a complex inner product space and let $a, b \in A$, $||a|| = ||b||$ and let α, β be any real scalars. Note that since α is real $(a, \alpha b) = \overline{(\alpha b, a)}$ $= \overline{\alpha(b,a)} = \overline{\alpha(b,a)} = \alpha(a,b)$. Thus

$$\begin{aligned} ||\alpha+\beta b||^2 &= (\alpha a+\beta b, \alpha a+\beta b) = \alpha^2||a||^2 + \alpha\beta(a,b) + \beta\alpha(b,a) \\ &+ \beta^2||b||^2 = \alpha^2||b||^2 + \alpha\beta(a,b) + \beta\alpha(b,a) \\ &+ \beta^2||a||^2 = (\alpha b+\beta a, \alpha b+\beta a) = ||\beta a+\alpha b||^2 \end{aligned}$$

Thus $||\alpha a+\beta b|| = ||\beta a+\alpha b||$. This proves the necessity of the condition.

Chapter 2

Some Geometric Characterizations

We proceed now to several characterizations of the norm in an inner product space which are geometrical in nature. The work in this chapter will be based on a paper by E. R. Lorch [21].

The first theorem is an apparent weakening of a characterization given by Ficken which we have called Theorem 1.4. We will show, however, that the hypothesized condition is equivalent to Ficken's.

Theorem 2.1: *Let* A *be a normed real linear space.* A *is an inner product space if and only if there exists a* fixed real constant, $\gamma \neq 0$, *such that* a, $b \in A$ *and* $||a|| = ||b||$ *imply* $||a+\gamma b|| = ||\gamma a+b||$.

Second formulation: By letting $a = c+d$, $b = c-d$, and $\gamma = \frac{1-\alpha}{1+\alpha}$, and noting that $||c+d + \frac{1-\alpha}{1+\alpha}(c-d)||$ $= ||\frac{1-\alpha}{1+\alpha}(c+d) + c-d||$ if and only if $||c+\alpha c+d+\alpha d+c-\alpha c-d+\alpha d|| = ||c-\alpha c+d-\alpha d+c+\alpha c-d-\alpha d||$ if and only if $||c+\alpha d|| = ||c-\alpha d||$, we see that an equivalent formulation of our theorem would be A is an inner product space if and only if there exists an $\alpha \neq 0, \pm 1$ such that $||c+d|| = ||c-d||$ implies $||c+\alpha d|| = ||c-\alpha d||$ for all c, $d \in A$.

<u>Proof of Theorem</u> 2.1. Let A be an inner product space. Then $||a+\gamma b||^2 = (a+\gamma b, a+\gamma b) = ||a||^2 + 2\gamma(a,b) + \gamma^2||b||^2$; and, if $||a|| = ||b||$, $||a+\gamma b||^2 = ||b||^2 + 2\gamma(a,b) + \gamma^2||a||^2 = ||\gamma a+b||^2$. Thus the condition of the theorem is necessary in an inner product space.

To prove the sufficiency of the condition we need the help of several lemmas. In each of these lemmas the condition stated in the <u>second formulation</u> of Theorem 2.1 is a tacit hypothesis.

<u>Lemma</u> 1. <u>If</u> $||a+b|| = ||a-b||$, <u>then</u> $||a+\gamma^n b|| = ||a-\gamma^n b||$, <u>where</u> n <u>is any integer</u>.

<u>Proof</u>. Let P(n) be the proposition that $||a+\gamma^n b|| = ||a-\gamma^n b||$. P(0) is hypothesized and P(1) is given by our tacit hypothesis.

If $||a+\gamma^n b|| = ||a-\gamma^n b||$ for some positive integer n, then the condition of the theorem gives us $||a+\gamma^{n+1}b|| = ||a-\gamma^{n+1}b||$. Thus P(n) is true for all positive integers and zero.

Now $||b+a|| = ||b-a||$ implies $||\gamma a+b|| = ||\gamma a-b||$ which gives in turn $||a+\gamma^{-1}b|| = a-\gamma^{-1}b||$. Thus P(n) is true for all integers n.

From the statement that $||a+b|| = ||a-b||$ implies $||a+\gamma^{-1}b|| = ||a-\gamma^{-1}b||$, we see that in the condition of the second formulation of our theorem we may require that $0 < \alpha < 1$.

Lemma 2. *If* α *is any real number, the function* $\phi(\alpha) = ||a+\alpha b||$ *is everywhere convex down*.

Proof. Choose α_1, α_2 with $\alpha_1 < \alpha_2$ and a real number t such that $0 < t < 1$. Let $\alpha = t\alpha_2 + (1-t)\alpha_1$ Then by the triangle inequality $||a+\alpha b|| = ||t(a+\alpha_1 b) + (1-t)(a+\alpha_2 b)|| \leqq t||a+\alpha_1 b|| + (1-t)||a+\alpha_2 b||$. Thus $\phi(\alpha) \leqq t\phi(\alpha_1) + (1-t)\phi(\alpha_2)$.

Lemma 3. *The function* $\phi(\alpha) = ||a+\alpha b||$ *is not constant on any interval*.

Proof. Note first that, since $||a+\alpha b|| > |\alpha|\cdot||b|| - ||a||$, $\lim_{\alpha\to\pm\infty} \phi(\alpha) = \infty$ except in the case in which $b = 0$. Note also that $\big|\, ||a+\alpha_1 b|| - ||a+\alpha_2 b||\,\big| \leqq |\alpha_1-\alpha_2|\cdot||b||$. Thus $\phi(\alpha)$ is continuous.

Suppose that $\phi(\alpha)$ is constant on some interval (δ, ϵ). By Lemma 2 $\phi(\alpha)$ is monotone non-decreasing for $\alpha > \epsilon$ and monotone non-increasing for $\alpha < \delta$. Since $\lim_{\alpha\to\pm\infty} \phi(\alpha) = \infty$, there is a maximum closed interval, call it $[\delta, \epsilon]$, over which $\phi(\alpha)$ is constant. Choose λ so that $\epsilon < \lambda < 2\epsilon-\delta$. Now if γ, $0 < \gamma < 1$, is the number from Lemma 1, $\eta = \gamma^n(\lambda-\delta)/2$ can, by choosing n sufficiently large, be made small emough that $\frac{\lambda+\delta}{2} \pm \eta$ are both contained in $[\delta, \epsilon]$. Since ϕ is assumed constant over $[\delta, \epsilon]$,

$\phi(\frac{\lambda+\delta}{2} + \eta) = \phi(\frac{\lambda+\delta}{2} - \eta)$, or, if we take $c = a + (\frac{\lambda+\delta}{2})b$, $d = \eta b$, $||c+d|| = ||c-d||$. By Lemma 1 then $||c+\gamma^s d|| = ||c-\gamma^s d||$ for any integer s. Choose $s = -n$ and we have $c + \gamma^{-n}d = a + (\frac{\lambda+\delta}{2})b + (\frac{\lambda-\delta}{2})b = a + \lambda b$ and $c - \gamma^{-n}d = a + (\frac{\lambda+\delta}{2})b - (\frac{\lambda-\delta}{2})b = a + \delta b$. Thus $\phi(\lambda) = \phi(\delta)$. Since $\lambda > \epsilon$, this violates the assumption that the interval of constancy $[\delta, \epsilon]$ is maximal. Thus ϕ is not constant over any interval.

Lemma 4. If $||a+b|| = ||a-b||$, then for all real numbers λ, $||a+\lambda b|| = ||a-\lambda b||$.

Proof. Again let $\phi(\lambda) = ||a+\lambda b||$. There is a γ, $0 < \gamma < 1$, such that $\phi(\gamma^n) = \phi(-\gamma^n)$, $n = 0, \pm 1, \pm 2, \ldots$. Suppose $\phi(\lambda) = \phi(-\lambda)$ is not true for all real numbers λ. Then there are two numbers $\alpha \neq 0$, $\beta \neq 0$, such that $\phi(\alpha+\beta) = \phi(\alpha-\beta)$. That is, $||a+(\alpha+\beta)b|| = ||a+(\alpha-\beta)b||$ or $||a+\alpha b)+\beta b|| = ||(a+\alpha b)-\beta b||$. By Lemma 1 $||a+\alpha b+\beta^s b|| = ||a+\alpha b-\beta^s b||$ for all integers s. Hence $\phi(\alpha+\beta^s) = \phi(\alpha-\beta^s)$. Suppose that $0 < \beta < 1$ (otherwise we may take s to be negative); then, since ϕ is continuous, $\lim_{s\to\infty} \phi(\alpha+\beta^s) = \lim_{s\to\infty} \phi(\alpha-\beta^s) = \phi(\alpha)$. Since ϕ is everywhere convex down, this means that α gives an absolute minimum for ϕ. But 0 is also a point of absolute minimum for ϕ. Since ϕ is convex down and not constant on any interval, this is a contradiction. Thus $\phi(\lambda) = \phi(-\lambda)$ for all real λ.

To prove Theorem 2.1, we see that Lemma 4 gives us that $||a+b|| = ||a-b||$ implies $||a+\lambda b|| = ||a-\lambda b||$ for all real scalars λ. Now let α, β be an real scalars, $\alpha + \beta \neq 0$, and let $c = a+b$, $d = a-b$. Choose $\lambda = \frac{\alpha-\beta}{\alpha+\beta}$. Then $||c|| = ||d||$ implies $||a+\lambda b|| = ||a-\lambda b||$, which in turn implies $||\alpha c+\beta b|| = ||\beta c+\alpha d||$. Thus the conditions of Theorem 1.4 are satisfied and A is a real inner product space.

A geometrical interpretation of the condition hypothesized in Theorem 2.1 is as follows: If for every line through the origin there exists a pair of points symmetric about the line such that their distances from the origin are equal, then the space is an inner product space.

Corollary 2.1.1. Let A be a normed real linear space. Then A is an inner product space if and only if $a, b, c \in A$, $a+b+c = 0$, and $||a|| = ||b||$ imply $||a-c|| = ||b-c||$. (Geometrically: A triangle is isosceles if and only if two medians are equal.)

[Note that the implication is symmetric; that is, $||c-a|| = ||b-c||$ implies $||a|| = ||b||$.

Proof: $(a-b) + (b-c) + (c-a) = 0$ and if the condition of the theorem holds, then $||a-c|| = ||b-c||$ implies $||c+b-2a|| = ||c+a-2b||$; but, since $a+b+c = 0$, this means $||a|| = ||b||$.]

<u>Proof of Corollary</u> 2.1.1. Suppose A is a real inner product space; $a,b,c \in A$; $a+b+c = 0$; and $||a|| = ||b||$. Then $||a-c||^2 + ||a-b||^2 = (a-c,a-c) + (a-b,a-b) = 2||a||^2 + ||c||^2 + ||b||^2 - 2(a,b+c) = 2||a||^2 + ||c||^2 + ||b||^2 - 2(a,-a) = 4||a||^2 + ||b||^2 + ||c||^2$. In the same manner $||b-c||^2 + ||a-b||^2 = 4||b||^2 + ||c||^2 + ||a||^2$. Since $||a|| = ||b||$, $||b-c||^2 + ||a-b||^2 = ||a-c||^2 + ||a-b||^2$, which is to say $||b-c|| = ||a-c||$. Now suppose A is a normed real linear space. Suppose further that $a,b,c \in A$; $a+b+c=0$; and $||a|| = ||b||$ imply $||a-c|| = ||b-c||$. Then $a-c = 2a+b$ and $b-c = a+2b$. Thus $||a|| = ||b||$ implies $||2a+b|| = ||a+2b||$. By Theorem 2.1, A is an inner product space.

<u>Corollary</u> 2.1.2. <u>Let</u> A <u>be</u> <u>a</u> <u>normed</u> <u>real</u> <u>linear</u> <u>space</u>. A <u>is</u> <u>an</u> <u>inner</u> <u>product</u> <u>space</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $a,b,c,d \in A$; $a+b+c+d = 0$; <u>and</u> $||a|| = ||b||$, $||c|| = ||d||$ <u>imply</u> $||a-c|| = ||b-d||$ <u>and</u> $||b-c|| = ||a-d||$.

<u>Proof</u>. Let A be a real inner product space and let $a,b,c,d, \in A$, $a+b+c+d = 0$, $||a|| = ||b||$ and $||c|| = ||d||$. Then $||a-c||^2 + ||a-b||^2 = 2||a||^2 + ||c||^2 + ||b||^2 - 2(a,b+c)$. $||b-c||^2 + ||a-b||^2 = 2||b||^2 + ||d||^2 + ||a||^2 - 2(b,a+d)$. Subtracting and recalling that $||a|| = ||b||$ and $||c|| = ||d||$, we get

$$||a-c||^2 - ||b-d||^2 = 2(b,a+d) - 2(a,b+c)$$
$$= 2(b,a+d) + 2(a,a+d) = 2(a+b,a+d).$$

Now $(a+b,a) = ||a||^2 + (b,a) = ||b||^2 + (a,b) = (a+b,b)$.

and $(a+b,d) = (-c-d,d) = -||d||^2 - (c,d) = -||c||^2 - (c,d) = -(c+d,c) = (a+b,c)$. Thus $(a+b,a+d) = (a+b,b+c) = -(a+b,a+d) = 0$. Therefore $||a-c|| = ||b-d||$. In a completely analogous manner it can be shown that $||b-c|| = ||a-d||$.

Now suppose that A is a normed real linear space; and, that $a,b,c,d \in A$, $a+b+c+d = 0$, $||a|| = ||b||$ and $||c|| = ||d||$ imply $||a-c|| = ||b-d||$ and $||b-c|| = ||a-d||$. Let $a' = a-d$ and $b' = b-c$, $c' = d-b$ and $d' = c-a$. Then $a'+b'+c'+d' = 0$ and $||a'|| = ||b'||$ and $||c'|| = ||d'||$. Applying the hypothesized condition to these vectors we get $||a'-d'|| = ||b'-c'||$ which is to say $||2a-c-d|| = ||2b-c-d||$. Since $a+b = -c-d$, we have $||3a+b|| = ||a+3b||$. Now let $a,b \in A$, $||a|| = ||b||$. Since $a+b+(-a)+(-b) = 0$ and $||-a|| = ||-b||$, the work just done shows that $||3a+b|| = ||a+3b||$. By Theorem 2.1, A is an inner product space.

Corollary 2.1.3. *Let A be a normed real linear space. A is an inner product space if and only if $a_1, a_2, b \in A$ imply*

$$\phi(a_1,a_2,b) = ||a_1+a_2+b||^2 + ||a_1+a_2-b||^2 - (||a_1-a_2-b||^2 + ||a_1-a_2+b||^2)$$

is independent of b.

Proof. Let A be a real inner product space with $a_1,a_2,b \in A$.

By the parallelogram law (Theorem 1.2)

$$\phi(a_1,a_2,b) = 2||a_1+a_2||^2 + 2||b||^2 - 2||a_1-a_2||^2 - 2||b||^2 = 2(||a_1+a_2||^2 - ||a_1-a_2||^2$$

which is independent of b.

Now suppose A is a normed real linear space in which $a_1,a_2,b \in A$ imply $\phi(a_1,a_2,b)$ is independent of b. Then $\phi(a_1,a_2,0) = \phi(a_1,a_2,a_2)$. That is, $||a_1+a_2||^2 + ||a_1+a_2||^2 - ||a_1-a_2||^2 - ||a_1-a_2||^2 = ||a_1-2a_2||^2 + ||a_1||^2 + ||a_1-2a_2||^2 - ||a_1||^2$. Thus $2||a_1+a_2||^2 - 2||a_1-a_2||^2 = ||a_1+2a_2||^2 - ||a_1-2a_2||^2$. Therefore $||a_1+a_2|| = ||a_1-a_2||$ implies $||a_1+2a_2|| = ||a_1-2a_2||$ and by Theorem 2.1, A is an inner product space.

<u>Theorem</u> 2.2. <u>Let</u> A <u>be</u> <u>a</u> <u>normed</u> <u>real</u> <u>linear</u> <u>space</u>. A <u>is</u> <u>an</u> <u>inner</u> <u>product</u> <u>space</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $a,b \in A$, $||a|| = ||b||$ <u>imply</u> $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$ <u>for</u> <u>all</u> <u>real</u> <u>numbers</u> $\alpha \neq 0$.

(Note: Since $||-b|| = ||b||$, the condition could also be written $||\alpha a-\alpha^{-1}b|| \geqq ||a-b||$.)

<u>Proof</u>. Let A be a real inner product space, $a,b \in A$, $||a|| = ||b||$. We note first that for any $\alpha \neq 0$, $\alpha^2 + \alpha^{-2} \geqq 2$. Expanding the norm in terms of inner products gives us $||\alpha a+\alpha^{-1}b||^2 = \alpha^2||a||^2 + 2(a,b) + \alpha^{-2}||b||^2 = (\alpha^2+\alpha^{-2})\frac{||a||^2+||b||^2}{2} + 2(a,b) \geqq ||a||^2 + ||b||^2 + 2(a,b) = ||a+b||^2$. Thus $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$.

Now suppose A is a normed real linear space in which $||a|| = ||b||$ and $\alpha \neq 0$ imply $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$. The proof that A is an inner product space will depend on the following lemmas.

Lemma 5. If A is a normed real linear space in which $a,b \in A$, $||a|| = ||b||$, and $\alpha \neq 0$ imply $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$, the inequality sign holds only if $\alpha = \pm 1$.

Proof. Suppose for $\alpha \neq \pm 1$, $||a+b|| = ||\alpha a+\alpha^{-1}b||$. Then by the triangle inequality

$$\left|\left| \left(\frac{1+\alpha}{2}\right)a + \left(\frac{1+\alpha^{-1}}{2}\right)b \right|\right| \leqq ||a+b||.$$

Since $(1-\alpha)^2 > 0$, and assuming, without loss of generality, that $\alpha > 0$ we have

$$\frac{1+\alpha}{2} \cdot \frac{1+\alpha^{-1}}{2} = \frac{(1+\alpha)^2}{4\alpha} > 1.$$

Let $\beta^2 = \frac{1+\alpha}{2} \cdot \frac{1+\alpha^{-1}}{2}$. Then

$$\left|\left|\left(\frac{1+\alpha}{2\beta}\right)a + \left(\frac{1+\alpha^{-1}}{2\beta}\right)b\right|\right| < ||a+b||.$$

But

$$\left|\left|\left(\frac{1+\alpha}{2\beta}\right)a + \left(\frac{1+\alpha^{-1}}{2\beta}\right)b\right|\right| = \left|\left|\left(\frac{1+\alpha}{1+\alpha^{-1}}\right)^{\frac{1}{2}}a + \left(\frac{1+\alpha^{-1}}{1+\alpha}\right)^{\frac{1}{2}}b\right|\right|$$

$$= \left|\left|\left(\frac{1+\alpha}{1+\alpha^{-1}}\right)^{\frac{1}{2}}a + \left(\frac{1+\alpha}{1+\alpha^{-1}}\right)^{-\frac{1}{2}}b\right|\right| \geqq ||a+b||$$

Since it is clear that equality does hold in the condition if $\alpha = \pm 1$, this contradiction proves the lemma.

Now let (*) denote the condition that $||a|| = ||b||$ and $\alpha \neq 0$ imply $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$ for all $a,b \in A$. Let (**) denote the converse of (*); i.e.,

(**): $a,b \in A$ and $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$ for all real $\alpha \neq 0$ imply $||a|| = ||b||$.

Lemma 6. *In a normed real linear space A (*) holds if and only if (**) holds.*

Proof. Suppose (*) holds in A. Choose $a,b \in A$ such that $||\alpha a+\alpha^{-1}b|| \geqq ||a+b||$ for all $\alpha \neq 0$. We shall show that $||a|| = ||b||$.

Clearly we may assume $a+b \neq 0$, for otherwise $||a|| = ||b||$. Suppose that $a \neq 0$, then b is necessarily different from 0. Let $\gamma = \dfrac{||b||}{||a||}$ and let $b = \gamma c$; then $||a|| = ||c||$. Using (*) we get $||a+b|| = ||a+\gamma c|| = \gamma^{\frac{1}{2}}||\gamma^{-\frac{1}{2}}a+\gamma^{\frac{1}{2}}c|| \geqq \gamma^{\frac{1}{2}}||a+c||$. By Lemma 5 the equality sign holds only if $\gamma = 1$, in which case $||a|| = ||b||$. By the hypothesis (*) $\gamma^{\frac{1}{2}}||a+c|| = ||\gamma^{\frac{1}{2}}a+\gamma^{\frac{1}{2}}c|| = ||\gamma^{\frac{1}{2}}a+\gamma^{-\frac{1}{2}}b|| \geqq ||a+b||$. Thus $\gamma^{\frac{1}{2}}||a+c|| = ||a+b||$, which means $||a|| = ||b||$.

Suppose now that A is a normed real linear space in which (**) holds and choose $a,b \in A$, $a,b \neq 0$ such that $||a|| = ||b||$. We have already seen in Lemma 3 earlier that the function $\theta(\alpha) = ||\alpha a+\alpha^{-1}b|| \doteq |\alpha|\cdot||\alpha^2a+b||$ is continuous (except at $\alpha = 0$) and assumes a minimum value at some point, say $\alpha = \alpha_o$. Since θ is symmetric, we may assume $\alpha_o > 0$. Then for all real $\alpha \neq 0$ $||\alpha a+\alpha^{-1}b|| \geqq ||\alpha_o a+\alpha_o^{-1}b||$. Thus $||(\alpha\alpha_o^{-1})\alpha_o a + (\alpha\alpha_o^{-1})^{-1}\alpha_o^{-1}b|| \geqq ||\alpha_o a+\alpha_o^{-1}b||$ for all values of $\alpha\alpha_o^{-1}$. Hence by (**) $||\alpha_o a|| = ||\alpha_o^{-1}b||$ which is to say $\alpha_o^2||a|| = ||b||$. Since $||a|| = ||b|| \neq 0$, $\alpha_o = 1$ and the lemma is proved.

We are now able to proceed with the proof of Theorem 2.2. Choose $a,b \in A$ such that $||a|| = ||b||$ and let θ be a real number. For any real ϕ $||e^{\phi}(e^{\theta}a+e^{-\theta}b) + e^{-\phi}(e^{-\theta}a+e^{-\theta}b)|| = ||(e^{\phi+\theta}+e^{-\phi-\theta})a + (e^{\phi-\theta}+e^{-\phi+\theta})b||$. Let $\alpha = e^{\phi+\theta} + e^{-\phi-\theta}$, $\beta = e^{\phi-\theta} + e^{-\phi+\theta}$. Then $\alpha\beta = e^{2\theta} + e^{-2\theta} + e^{2\phi} + e^{-2\phi} \geqq (e^{\theta}+e^{-\theta})^2$, since $e^{2\phi} + e^{-2\phi} \geqq 2$.

Let $\gamma^2 = \alpha\beta$. Using (*) we get $||\alpha a+\beta b|| = \gamma||\alpha\gamma^{-1}a+\beta\gamma^{-1}b|| = \gamma||(\alpha/\beta)^{\frac{1}{2}}a+(\alpha/\beta)^{-\frac{1}{2}}b|| \geqq \gamma||a+b|| \geqq (e^{\theta} + e^{-\theta})||a+b|| = ||(e^{\theta}a+e^{-\theta}b) + (e^{-\theta}a+e^{\theta}b)||$. Thus $||e^{\phi}(e^{\theta}a+e^{-\theta}b) + e^{-\phi}(e^{-\theta}a+e^{\theta}b)|| \geqq ||(e^{\theta}a+e^{-\theta}b) + (e^{-\theta}a+e^{\theta}b)||$ for all real ϕ. Using Lemma 6, we conclued that $||e^{\theta}a+e^{-\theta}b|| = ||e^{-\theta}a+e^{\theta}b||$. That is, $||e^{2\theta}a+b|| = ||a+e^{2\theta}b||$. By Theorem 2.1, A is an inner product space.

The following theorem is as stated and proved by E. R. Lorch [21] in 1948. The same result was given by N. Aronszajn ([1], page 811 and [1], page 874, Theorem IV) in 1935, but no proof was provided in these references. The geometrical meaning of the condition is simply that the length of the median of a triangle is completely determined by the lengths of the sides of the triangle.

<u>Theorem</u> 2.3.([21], page 524) <u>A normed real linear space A is an inner product space if and only if there is a non-trivial functional relation</u> $F(x,y,u,v) = 0$ <u>such that</u> a <u>and</u> $b \in A$ <u>imply that this relation is satisfied by</u> $x = ||a||$, $y = ||b||$, $u = ||a+b||$, $v = ||a-b||$.

<u>Proof</u>. The necessity of the condition follows directly from the parallelogram law (Theorem 1.2). Let $a, b \in A$, then let $\phi(x,y,u,v) = 2x^2 + 2y^2 - u^2 - v^2$. The parallelogram law tells us that the equation $\phi(x,y,u,v) = 0$ is satisfied by $x = ||a||$, $y = ||b||$, $u = ||a+b||$, $v = ||a-b||$.

To prove the sufficiency of the condition, consider $a, b \in A$ with $||a|| = ||b||$. Write $a' = a$, $a'' = b$, $b' = b'' = a+b$. Then $b'-a' = b$ and $b''-a'' = a$. Thus $||a'|| = ||a''||$, $||b'|| = ||b''||$, and $||b'-a'|| = ||b''-a''||$. Applying the condition of the theorem and using the non=trivial nature of F, we have

$F(||a'||,||b'||,||b'-a'||,||a'+b'||) =$

$F(||a'||,||b'||,||b'-a'||,||a''+b''||) = 0$, leading to $||a'+b''|| = ||a''+b''||$. This gives us $||2a+b|| = ||a+2b||$. By Theorem 2.1 A is an inner product space.

One additional result given by Lorch will be presented without proof. The proof is given in considerable detail by Lorch in [21], pages 530-532.

Theorem 2.4. *A normed real linear space A is an inner product space if and only if for a fixed integer $n \geq 3$ $a_1, \ldots, a_n \in A$ and $a_1 + \cdots + a_n = 0$ imply*

$$\sum_{i \neq j} ||a_i - a_j||^2 = 2n \sum_i ||a_i||^2.$$

General Corollary. *The conditions given in Theorems 2.1, 2.2, and 2.3 and in Corollaries 2.1.1, 2.1.2, and 2.1.3 are also necessary and sufficient conditions that a normed complex linear space be an inner product space.*

Proof. The necessity of these conditions in a complex inner product space is easily seen by simply expanding the norms in terms of complex inner products.

For the sufficiency of the conditions we employ Theorem 1.3. We let A be a complex linear space which exhibits any of these characteristics and let B be its associated real linear space. B is necessarily an inner product space as we have shown here and so, by Theorem 1.3, is A.

Chapter 3

A Weakening of the Parallelogram Law

The main theorem of this chapter represents the latest in a chain of successive weakenings of the conditions of the parallelogram law. M. M. Day first proved that if the parallelogram law holds for points on the unit sphere of a normed linear space, then the space must be an inner product space. We shall refer to this as condition (D). (See [5], Theorem 2.1.)

(D) $a,b \in A$ and $||a|| = ||b|| = 1$ imply

$||a+b||^2 + ||a-b||^2 = 4.$

I. J. Schoenberg, using Day's technique, further weakened the condition by noting that the equal sign could be replaced by either $\geqq$ or $\leqq$, the same sign being used throughout. (See [29] Theorem 1).

(s,$\sim$) $a,b \in A$ and $||a|| = ||b|| = 1$ imply $||a+b||^2 + ||a-b||^2 \sim 4$, where for the symbol $\sim$ may be read $\geqq$, $=$, or $\leqq$, the same sign being used throughout the argument.

S. K. Kasahara suggested a new approach with the following necessary and sufficient condition. (See [18] or [7], page 93.)

(K) There exists a positive number $\alpha \leqq \frac{1}{2}$ and for each pair $a,b \in A$ there is a number β, depending on a and b, such that $\alpha \leqq \beta \leqq 1-\alpha$ and $\beta||a||^2 + (1-\beta)||b||^2 \geqq \beta(1-\beta)||a-b||^2 + ||\beta a+(1-\beta)b||^2$.

Day returned to the scene and weakened Kasahara's hypothesis by dropping the uniformity imposed by the existence of the number α, by restricting consideration to points on the unit sphere, and by showing that $\geqq$ could be replaced by $=$ or $\leqq$. (See [7] Theorem 1.) Day's result is our theorem.

Theorem 3.1. Let A be a normed real linear space. A is an inner product space if and only if for each pair $a,b \in A$ with $||a|| = ||b|| = 1$, there exist α and β, $0 < \alpha < 1$, $0 < \beta < 1$, such that $(\alpha+\beta-2\alpha\beta)(\alpha\beta+(1-\alpha)(1-\beta)) \geqq \beta(1-\beta)||\alpha a+(1-\alpha)b||^2 + \alpha(1-\alpha)||\beta a+(1-\beta)b||^2$. Additional necessary and sufficient conditions are given when $\geqq$ is replaced by $=$ or by $\leqq$, the same sign being used throughout the argument.

Proof. Necessity of the condition is proved in the conventional manner. Let A be a real inner product space. Then for $a,b \in A$ with $||a|| = ||b|| = 1$ and for scalars α and β we have $||\alpha a+(1-\alpha)b||^2 = \alpha^2+2\alpha(1-\alpha)(a,b)+(1-\alpha)^2$ and $||\beta a+(1-\beta)(-b)||^2 = \beta^2-2\beta(1-\beta)(a,b)+(1-\beta)^2$.

Eliminating (a,b) from the above equations we get $\beta(1-\beta)||\alpha a+(1-\alpha)b||^2 + \alpha(1-\alpha)||\beta a-(1-\beta)b||^2 = \alpha^2\beta(1-\beta) + \beta^2\alpha(1-\alpha) + (1-\alpha)^2\beta(1-\beta) + (1-\beta)^2\alpha(1-\alpha) = [\alpha(1-\beta) + \beta(1-\alpha)][\alpha\beta+(1-\alpha)(1-\beta)] = [\alpha+\beta-2\alpha\beta][\alpha\beta+(1-\alpha)(1-\beta)]$.
We see therefore that in an inner product space equality must hold in the condition, so $\geq$ or $\leq$ are also necessary.

The sufficiency of the condition will be proved with the aid of two lemmas which, although they appear in [5], page 322, are attributed by Day to some unpublished work of Loewner.

Lemma 1. If C is a symmetric closed convex curve (on the real plane), then there exists a unique ellipse of minimal area circumscribed about C.

Proof. Let E_1 amd E_2 be two ellipses of minimal area about the symmetric closed convex curve C. Make an affine transformation reducing the ellipses to principal axes so that equations for them may be written in the form $x^2 + y^2 = 1$ and $ax^2 + by^2 = 1$. Since these ellipses are of equal area π, $ab = 1$. If (x,y) lies on C, then $x^2+y^2 \leq 1$ and $ax^2+by^2 \leq 1$ since E_1 and E_2 both circumscribe C. Thus $\frac{(a+1)}{2}x^2 + \frac{(b+1)}{2}y^2 \leq 1$; that is, the ellipse $\frac{(a+1)}{2}x^2 + \frac{(b+1)}{2}y^2 = 1$ also encloses C. The area of this last ellipse is $\frac{2\pi}{\sqrt{(a+1)(b+1)}}$, which is greater than π unless $a = b = 1$.

Thus E_1 and E_2 are the same ellipses.

Lemma 2. The (unique) minimal ellipse circumscribed about the symmetric closed curve C touches C in at least four points.

Proof. Arrange the coordinate system so that the minimal ellipse E has the equation $x^2 + y^2 = 1$, where $(1,0)$ is a point of contact of C and E. The ellipses E_n with equations $\frac{x^2}{(1+\frac{1}{n})} + (1+\frac{1}{n})y^2 = 1$ have the minimal area and must therefore not enclose C. For each integer n let p_n be a point of C not in E_n. Then there is a sequence n_k such that p_{n_k} converges to some point p. Since the p_n are all on C, p is on C. p_n is within $\frac{1}{n}$ of E, so p is also on E. The intersections of E and E_n have y coordinates $\pm(2+\frac{1}{n})^{-\frac{1}{2}}$, hence the absolute value of the y coordinate of p is at least $(2)^{-\frac{1}{2}}$. p is not therefore the point $(1,0)$ or the point $(-1,0)$, but p is on both E and C. By symmetry of E and C, so is -p. Thus C and E have at least four points in common.

Note: These two lemmas have complete analogues if the words "minimal" and "circumscribed" are replaced by "maximal" and "inscribed". The proofs are carried out in a completely analogous manner. We shall use these analogues as well as the original lemmas in our succeeding proof.

The proof of the sufficiency of the condition in our theorem now follows quickly. Let A be a normed real linear space in which the condition holds. Let S be any two-dimensional subspace of A. We shall show that C, the set of elements of S having norm one, is an ellipse.

Let E be the ellipse of minimal area circumscribed about C and let $|\cdot|$ designate the norm for which E is the boundary of the unit sphere. By Lemma 2 E touches C at at least four distinct points $\pm a$, $\pm b$. Let K be the set of all points common to C and E. Since K is clearly a closed subset of C and E, L, the complement of K in E, is open in E.

If L is not empty, consider an open arc M in L with end points at c and d in K. (Note that, by Lemma 2, $c \neq \pm d$.) Since S is an inner product space with the norm $|\cdot|$, the necessity of our condition in inner product spaces tells us that the α, β, $0 < \alpha < 1$, $0 < \beta < 1$, given by our condition are such that $(\alpha+\beta-2\alpha\beta)[\alpha\beta+(1-\alpha)(1-\beta)]$ $= \beta(1-\beta)|\alpha c+(1-\alpha)d|^2+\alpha(1-\alpha)|\beta c-(1-\beta)d|^2$. Since E is everywhere outside C, $||f|| \geq |f|$ for all $f \in S$. Therefore we have $(\alpha+\beta-2\alpha\beta)[\alpha\beta+(1-\alpha)(1-\beta)] = \beta(1-\beta)|\alpha c+(1-\alpha)d|^2$ $+ \alpha(1-\alpha)|\beta c-(1-\beta)d|^2 \leq \beta(1-\beta)||\alpha c+(1-\alpha)d||^2 + \alpha(1-\alpha)||\beta c-(1-\beta)d||^2 \leq$ $(\alpha+\beta-2\alpha\beta)[\alpha\beta+(1-\alpha)(1-\beta)]$. Since the ends are equal, we see that $\beta(1-\beta)|\alpha c+(1-\alpha)d|^2 + \alpha(1-\alpha)|\beta c-(1-\beta)d|^2 =$ $\beta(1-\beta)||\alpha c+(1-\alpha)d||^2 + \alpha(1-\alpha)||\beta c-(1-\beta)d||^2$. Since the scalars α, β, $1-\alpha$, $1-\beta$ are positive and $||f|| \geqq |f|$ for all $f \in S$, it follows that $|\alpha c+(1-\alpha)d| = ||\alpha c+(1-\alpha)d||$. But $\alpha c + (1-\alpha)d \neq 0$ since $c \neq \pm d$ and $||c|| = ||d||$. Therefore $\dfrac{\alpha c+(1-\alpha)d}{||\alpha c+(1-\alpha)d||}$ belongs to M and to K. Thus L must be empty; that is, $||f|| = |f|$ for all $f \in S$. By Theorem 1.1, S is an inner product space; and, by Corollary 1.4.1, so is A.

It is clear that replacing $\geqq$ by $\leqq$ or = in the the statement of the theorem and using the idea of maximal inscribed rather than minimal circumscribed ellipses proves the further assertion of our theorem.

We now investigate some of the immediate consequences of Theorem 3.1.

<u>Corollary</u> 3.1.1. <u>A real normed linear space</u> A <u>is an inner product space if and only if any one of the following conditions hold</u>. (It shall be understood, as before, that the use of the symbol $\sim$ shall mean that, when $\sim$ is replaced by $>$, $=$, or $<$, the statement derived is a characterization of inner product spaces.)

$(1,\sim)$ <u>For each pair</u> a <u>and</u> b <u>of points of</u> A <u>with</u> $||a|| = ||b|| = 1$ <u>there is a number</u> α <u>depending on</u> a <u>and</u> b <u>such that</u> $0 \leqq \alpha < 1$ <u>and</u>

$$1 \sim \alpha(1-\alpha)||a-b||^2 + ||\alpha a+(1-\alpha)b||^2.$$

(2,~) <u>For each pair</u> a <u>and</u> b <u>of points of</u> A <u>with</u> $||a|| = ||b|| = 1$ there exists α, $0 < \alpha < 1$, <u>such that</u>

$$2[\alpha + (1-\alpha)^2] \sim ||\alpha a+(1-\alpha)b||^2 + ||\alpha a-(1-\alpha)b||^2.$$

(3,~) <u>For each pair</u> a <u>and</u> b <u>of points of</u> A <u>with</u> $||a|| = ||b|| = 1$ <u>there exists</u> α <u>such that</u>

$$2(1-2\alpha(1-\alpha)) \sim ||\alpha a+(1-\alpha)b||^2 + ||(1-\alpha)a-\alpha b||^2.$$

<u>Proof</u>. The necessity of each of these conditions in an inner product space can be established by using the parallelogram law and choosing α to be the proper constant in each case. The details will not be discussed here.

The sufficiency of (1,~) follows from choosing β of Theorem 3.1 equal to the α given by the condition and α of Theorem 3.1 equal to $\frac{1}{2}$ for a and b in A.

In (2,~) choose both the α and the β of Theorem 3.1 to be the α provided in the condition, for all a and b in A.

In (3,~) sufficiency is established by choosing α of Theorem 3.1 to be the α provided by the condition and letting β of Theorem 3.1 equal $1-\alpha$.

Schoenberg's conditions, (S,~), and therefore Day's original condition (D), are easily seen to be special cases of (1,~) by choosing α of (1,~) to be $\frac{1}{2}$.

In the same manner the sufficiency of Kasahara's hypothesis (K) follows by restricting consideration to points on the unit sphere and choosing the α of $(1,\sim)$ to be the β given by (K).

Schoenberg used $(S,\geqq)$ to prove that a real ptolemaic semi-normed linear space is an inner product space ([29], Theorem 1).

Definition. *A semi-norm on a linear space* A *is a function* $||\cdot||: A \to$ *non-negative reals such that*

1. $||\alpha a|| = |\alpha|\cdot||a||$ *for all scalars* α *and all* $a \in A$.
2. $||a|| \geqq 0$ *for all* a *and* $||a|| = 0$ *if and only if* $a = 0$.

Definition. *A semi-normed space* A *is called ptolemaic if and only if* $||a-b||\cdot||c-d|| + ||a-d||\cdot||b-c|| \geqq ||a-c||\cdot||b-d||$ *for all* a,b,c,d *belonging to* A.

Corollary 3.1.2. *Let* A *be a ptolemaic semi-normed real linear space. Then the semi-norm on* A *is a norm which is given by an inner product on* A.

Proof. Let $f,g \in A$. Choose $a = 0$, $b = f$, $c = \frac{f+g}{2}$, $d = g$. Since A is ptolemaic

$$||f||\,\left|\left|\frac{f-g}{2}\right|\right| + ||g||\,\left|\left|\frac{f-g}{2}\right|\right| \geqq \left|\left|\frac{f+g}{2}\right|\right|\,||f-g||.$$

If $f \neq g$, divide by $||\frac{f-g}{2}||$ to get $||f|| + ||g|| \geqq ||f+g||$. Thus the semi-norm on A is a norm. Now let $f,g \in A$ with $||f|| = ||g|| = 1$. Let $a = f$, $b = g$, $c = -f$, $d = -g$ and apply the ptolemaic principle again to get

$$||f-g||\cdot||f-g|| + ||f+g||\cdot||f+g|| \geqq 4||f||\cdot||g|| = 4.$$

By $(1,\geqq)$ of Corollary 3.1.1, or in particular by $(S,\geqq)$, A is an inner product space.

Corollary 3.1.3. If A is a complex normed linear space, A is an inner product space if and only if for any pair $a,b \in A$ with $||a|| = ||b|| = 1$ there exist real scalars α,β, $0 < \alpha < 1$, $0 < \beta < 1$, such that

$$(\alpha+\beta-2\alpha\beta)[\alpha\beta+(1-\alpha)(1-\beta)] \geqq \beta(1-\beta)||\alpha a+(1-\alpha)b||^2 + \alpha(1-\alpha)||\beta a+(1-\beta)b||^2.$$

Again, additional necessary and sufficient conditions are given when $\geqq$ is replaced by $=$ or $\leqq$, the same sign being used throughout the argument.

Proof. Necessity can be established easily by expanding the norm in terms of inner products. To prove sufficiency let A be a complex normed linear space in which the condition holds and let B be its associated real linear space. B also enjoys the property hypothesized by the theorem; thus, by Theorem 3.1, B is an inner product space. By Theorem 1.3 A is also an inner product space.

Day([5] Theorem 4.1) used a method of proof similar to that used in Theorem 3.1 to prove the following characterization of inner product spaces among uniformly convex linear spaces.

Definition. A normed linear space A is called uniformly convex if and only if for each ϵ, $0 < \epsilon \leq 2$, there is a $\delta(\epsilon) > 0$ such that $||a+b|| \leq 2(1-\delta(\epsilon))$ if $a, b \in A$, $||a-b|| \geq \epsilon$, and $||a|| = ||b|| = 1$. The function $\delta(\epsilon)$ is called the modulus of convexity of A.

It is well known that every inner product space is uniformly convex. Clarkson, who is responsible for the concept of uniform convexity [4], showed that, if $\delta_2(\epsilon) = 1 - (1-(\frac{\epsilon}{2})^2)^{\frac{1}{2}}$, then $\delta_2(\epsilon)$ is the largest possible modulus of convexity for an inner product space. Day's theorem below will show that having a modulus of convexity no smaller than $\delta_2(\epsilon)$ will force the uniformly convex space to be an inner product space. Thus $\delta_2(\epsilon)$ is as big as a modulus of convexity can be.

Theorem 3.2. A uniformly convex normed real linear space A is an inner product space if and only if the modulus of convexity, $\delta(\epsilon)$ satisfies $\delta(\epsilon) \geq \delta_2(\epsilon)$ for all ϵ such that $0 < \epsilon < 2$. (Clarkson's result shows that δ is actually equal to δ_2.)

Proof. Necessity of the condition has been established by Clarkson in the work mentioned above.

To prove sufficiency let B be a two-dimensional subspac e of A and let c be the set of all points of norm one in B. As in the proof of Theorem 3.1, let E be the ellipse of smallest area circumscribed about C and $|\cdot|$ represent the norm in B for which E is the unit sphere. If a and b, $a \neq \pm b$, are points of contact of C and E, let $\epsilon_1 = |a-b|$ and $\epsilon_2 = ||a-b||$. Then $\epsilon_2 \geqq \epsilon_1$ and $2(1-\delta_2(\epsilon_1)) =$ $2[1-(\frac{|a-b|}{2})^2]^{\frac{1}{2}} = 2|a+b| \leqq 2||a+b|| \leqq 2(1-\delta(\epsilon_2)) \leqq$ $2(1-\delta_2(\epsilon_2)) \leqq 2(1-\delta_2(\epsilon_1))$. Since the ends are equal $|a+b| =$ $||a+b||$ and $|a-b| = \epsilon_1 = \epsilon_2 = ||a-b||$. Pursuing procedures used in the proof of Theorem 3.1, we can easily show that C is E and therefore, by Theorem1.1. and Corollary 1.4.1, A is an inner product space.

Chapter 4

Orthogonality and Orthogonal Projections

1. Preliminaries

In this chapter we shall consider various types of orthogonality in normed linear spaces and the properties of orthogonality that occur if and only if the normed linear space is an inner product space. In section five of this chapter we shall discuss a notion closely allied with the idea of an orthogonal projection in an inner product space.

The four types of orthogonality we shall discuss are:

1. Isosceles Orthogonality

   a is isosceles orthogonal to b if and only if $||a+b|| = ||a-b||$.

2. Pythagorean Orthogonality

   a is Pythagorean orthogonal to b if and only if $||a||^2 + ||b||^2 = ||a-b||^2$.

3. Normality

   a is normal to b if and only if $||a+\alpha b|| \geq ||a||$ for all real scalars α.

4. Euclidean Orthogonality

   In an inner product space we say a is Euclidean orthogonal to b if and only if $(a,b) = 0$.

We shall, in the first four sections of this chapter, restrict our considerations to normed real linear spaces.

By simply expanding the norms as inner products it is seen that in an inner product space isosceles and Pythagorean orthogonality are equivalent to Euclidean orthogonality. To see that normality is equivalent to Euclidean orthogonality in an inner product space, let A be an inner product space and let a and b belong to A, a normal to b. Then $||a+\alpha b|| \geq ||a||$ for all scalars α; that is, $||a||^2 + 2\alpha(a,b) + \alpha^2||b||^2 \geq ||a||^2$. Thus $2\alpha(a,b) + \alpha^2||b||^2 \geq 0$ for all α. Suppose $(a,b) \neq 0$; choose $\alpha = \frac{-(a,b)}{||b||^2}$. Then $2\alpha(a,b) + \alpha^2||b||^2 = -\frac{2(a,b)^2}{||b||^2} + \frac{(a,b)^2}{||b||^2} < 0$, contradicting the previous inequality. Thus $(a,b) = 0$. Furthermore, if $(a,b) = 0$, then for any α

$$||a+\alpha b||^2 = ||a||^2 + 2\alpha(a,b) + \alpha^2||b||^2 = ||a||^2 + \alpha^2||b||^2 \geq ||a||^2.$$ That is, $||a+\alpha b|| \geq ||a||$.

For brevity we shall use the symbol $\perp$ to indicate orthogonality whenever the context will indicate the type of orthogonality we mean.

We shall wish to discuss these various types of orthogonality in terms of four familiar geometric properties of Euclidean orthogonality.

1. Symmetry - $a \perp b$ implies $b \perp a$.
2. Homogeneity - $a \perp b$ implies $\alpha a \perp b$.
3. Additivity - $a \perp b$ and $a \perp c$ imply $a \perp (b+c)$.
4. Non-vacuous - For any a,b there is a scalar α such that $a \perp (a + \alpha b)$. (For Euclidean orthogonality the α is unique.)

It is easy to see that isosceles and Pythagorean are symmetric, but we shall find that, in normed linear spaces of dimension greater than two, the normality is symmetric only if the space is an inner product space. We shall also see that homogeneity and additivity of isosceles or Pythagorean orthogonality occur only in inner product spaces.

2. Isosceles Orthogonality

Theorem 4.1 ([12], Theorem 4.7): If isosceles orthogonality is homogeneous in a normed linear space A, then A is an inner product space.

Proof: Choose $a,b \in A$ such that $||a|| = ||b||$. Then $||(a+b) + (a-b)|| = ||(a+b) - (a-b)||$. That is $(a+b) \perp (a-b)$. If isosceles orthogonality is homogeneous in A, then

$$||(\alpha+1)(a+b) + (\alpha-1)(a-b)|| = ||(\alpha+1)(a+b) - (\alpha-1)(a-b)||$$

for all α. Thus $||\alpha a+b|| = ||a+\alpha b||$. By Theorem 1.4, A is an inner product space.

<u>Theorem 4.2</u> ([12], Theorem 4.8): <u>If isosceles orthogonality is additive in a normed linear space A, then A is an inner product space.</u>

<u>Proof</u>: If $a, b \in A$ and $a \perp b$, then $a \perp -b$ and $b \perp a$ because of the nature of isosceles orthogonality. If isosceles orthogonality is additive, then $na \perp mb$ for all integers m and n. Thus $||na+mb|| = ||na-mb||$. That is, $||a+m/nb|| = ||a-m/nb||$. Since the norm is continuous, it follows that $||a+\alpha b|| = ||a-\alpha b||$ for all real scalars α. This means that isosceles orthogonality is homogeneous if it is additive. By Theorem 4.1, A is an inner product space.

3. <u>Pythagorean Orthogonality</u>

Theorem 4.3 ([25]): <u>If Pythagorean orthogonality is homogeneous in a normed linear space A, then A is an inner product space.</u>

<u>Proof</u>: Let $a, b \in A$ and assume $||a|| \geq ||b||$. Let $f(\alpha) = ||a-(b+\alpha a)||^2 - ||a||^2 - ||b+\alpha a||^2$, then f is continuous.

i) Suppose $f(0) = ||a-b||^2 - ||a||^2 - ||b||^2 \geq 0$, then, since $||a|| \geq ||b||$ implies

$||a+b||^2 + ||a||^2 \geq 2||a||^2 - 2||a||\,||b|| + ||b||^2 \geq ||b||$

$f(1) = ||b||^2 - ||a||^2 - ||a+b||^2 \leq 0$. Thus there is a number α, $0 \leq \alpha \leq 1$, such that $f(\alpha) = 0$.

ii) Suppose $f(0) < 0$. If then $f(-1) \geq 0$, there is a number α, $-1 \leq \alpha < 0$ such that $f(\alpha) = 0$. If on the other hand $f(-1) = ||2a-b||^2 - ||a||^2 - ||a-b||^2 < 0$, then $f(-2) = ||3a-b||^2 - ||a||^2 - ||2a-b||^2$

$\geq ||3a-b||^2 - 2||a||^2 - ||a-b||^2 \geq ||3a-b||^2 - 3||a||^2 - ||b||^2$

$\geq ||3a||^2 + ||b||^2 - 2||3a||\ ||b|| - 3||a||^2 - ||b||^2$

$= 6||a||\ (||a|| - ||b||) \geq 0.$

Thus there is a number α, $-2 \leq \alpha < 1$ such that $f(\alpha) = 0$. Thus f has at least one zero.

Now let α be a zero of f. Let $c = b+\alpha a$. Then $b = c-\alpha a$ and $||a||^2 + ||c||^2 = ||a-c||^2$. Homogeneity of Pythagorean orthogonality gives us $\alpha^2||a||^2 + ||c||^2 = ||\alpha a-c||^2 = ||b||^2$ for all α. From this we get

$$||a-b||^2 + ||a+b||^2 = ||(1+\alpha)a-c||^2 + ||(1-\alpha)a+c||^2$$
$$= (1+\alpha)^2||a||^2 + ||c||^2 + (1-\alpha)^2||a||^2 + ||c||^2$$
$$= 2||a||^2 + 2(\alpha^2||a||^2 + ||c||^2)$$
$$= 2||a||^2 + 2||b||^2 .$$

Since a and b were arbitrary, the parallelogram law holds and A is an inner product space.

James shows ([12], Theorem 5.1) that, given any two elements, a and b, of a normed linear space, there is a scalar α such that a is Pythagorean orthogonal to $\alpha a-b$. We shall use this result to prove our next theorem.

Theorem 4.4 ([12], Theorem 5.3): If Pythagorean orthogonality is additive in a normed linear space A, then A is an inner product space. (Thus homogeneity and additivity of Pythagorean orthogonality are equivalent.)

Proof: Suppose Pythagorean orthogonality is additive in a normed linear space A. Let a and b be two orthogonal elements of A. The result of James cited above gives us a scalar α such that $a \perp (\alpha a - b)$. Additivity gives $a \perp \alpha a$, and hence $\alpha = 0$ if $a \neq \Theta$. Thus $a \perp -b$. Also by symmetry $b \perp a$. By additivity we get $na \perp mb$ for all integers m and n. Thus $||na||^2 + ||mb||^2 = ||na-mb||^2$ or

$||a||^2 + ||\frac{m}{n} b||^2 = ||a - \frac{m}{n} b||^2$. From the continuity of the norm it follows that $||a||^2 + ||\beta b||^2 = ||a-\beta b||^2$ for all real β. Thus Pythagorean orthogonality is homogeneous if it is additive. By Theorem 4.3, A is an inner product space.

M. M. Day proved that if isosceles orthogonality implies Pythagorean orthogonality or if Pythagorean implies isosceles orthogonality in a normed linear space, then the space is an inner product space. Since in an inner product space both Pythagorean and isosceles orthogonality are equivalent to Euclidean orthogonality, this gives us two more characterizations of inner product spaces.

Theorem 4.5 (5 , Theorem 5.1): *If isosceles orthogonality implies Pythagorean orthogonality in a normed linear space A, then A is an inner product space.*

Proof: Let $a,b \in A$, $||a|| = ||b|| = 1$. Then $a+b$ and $a-b$ are isosceles orthogonal. By hypothesis they are also Pythagorean orthogonal and $4 = ||2b||^2 = ||(a+b) - (a-b)||^2 = ||a+b||^2 + ||a-b||^2$. By (1,=) of Corollary 3.1 with $a = \frac{1}{2}$, A is an inner product space.

Theorem 4.6 ([5], Theorem 5.2): *If Pythagorean orthogonality implies isosceles orthogonality in a normed linear space A, then A is an inner product space.*

Proof: The proof will not be given in full detail. If a and b belong to A and a is Pythagorean orthogonal to b, then $||a-b||^2 = ||a||^2 + ||b||^2$. We see that a is isosceles orthogonal to b if and only if $||a+b|| = ||a-b||$, which is to say if and only if a is Pythagorean orthogonal to $-b$. Day then shows ([5], Lemma 5.3) that A is an inner product space if and only if a Pythagorean orthogonal to b implies a Pythagorean orthogonal to $-b$. The argument used involves the maximum modulus of convexity discussed in Theorem 3.2.

4. Normality

Unlike isosceles and Pythagorean orthogonality, normality is not necessarily symmetric. Birkhoff ([2], Theorem 1) proved that if normality is symmetric and for any a and b there is a unique α such that $a \perp (a+\alpha b)$ in a normed linear space A of dimension greater than two, then A is an inner product space. Birkhoff also showed that there are infinitely many metrically different two-dimensional linear spaces in which normality can be symmetric and unique ([2], Theorem 4). James ([13], Theorem 1) and Day ([5], Theorem 6.4) independently proved that normality is symmetric in a normed linear space of dimension greater than 2 only if the space is an inner product space. We shall display the proof given by James.

It will first be necessary to give, without proof, two theorems essential for our method of proof. The first of these was proved by Kakutani ([16]) and an alternated proof was given by Phillips ([26]).

Theorem 4.7: *A (real or complex) normed linear space A of dimension greater than 3 is an inner product space if and only if for every two dimensional closed linear subspace B of A there is a projection of norm one from A onto B.*

The second result we shall use is by James ([14], Theorem 7.1).

Lemma : *A necessary and sufficient condition that there exist an element normal to each closed linear subspace of a normed linear space A is that for each linear* *functional f defined on A there is an element $a \in A$ such that $f(a) = ||f|| \, ||a||$.*

Theorem 4.8: *Normalty is symmetric in a normed linear space A of three or more dimensions if and only if A is an inner product space.*

Proof: In an inner product space normality is equivalent to Euclidean orthogonality and is therefore symmetric.

Now let A be a normed linear space of dimension greater than two in which normality is symmetric. We may assume without loss of generality that the dimension of A is three. Let a,b be any independent elements of A and let H be the linear hull of a and b. Because every finite dimensional normed linear space is reflexive, the Lemma cited above gives us $c \in A$ such that $c \perp H$ (that is, c is normal to every element in H). By symmetry of normality, $H \perp c$. Hence if a projection P of A on B is defined by $d = P(d) + \alpha_d c$ where $P(d) \in H$ for all $d \in A$, then $||d|| = ||P(d) + \alpha_d c|| \geq ||P(d)||$ for all d. Since $P(d) = d$ if d belongs to H, $||P|| = 1$. This gives us a projection of norm one on the arbitrary closed linear subspace H. By Theorem 4.7, A is an inner product space.

Theorem 4.9: *A normed linear space A of dimension greater than two is an inner product space if and only if normality is additive on the left. That is, if and only if $a \perp c$ and $b \perp c$ implies $(a+b) \perp c$.*

Proof: Necessity of the condition in an inner product space is clear.

For proof of sufficiency, let A be a normed linear space of simension greater than 2 in which normality is additive on the left. Let a and b be any two elements of A. Then there are maximal linear subspaces of A, call them G and H, with $a \in G$ and $b \in H$. Let $M = G \cap H$. Since normality is additive on the left, $\alpha a + \beta b$ is normal to M for all α and β, and any element c has a unique representation in the form $c = P(c)+d$ where $d \in M$ and $P(c) = \alpha a+\beta b$. $||c|| = ||P(c)+d|| \geq ||P(c)||$ for all c in A, thus $||P|| = 1$. This gives us a projection of norm one on any two-dimensional linear subspace of A. Thus A is an inner product space.

Further theorems concerning properties of normality which characterize inner product spaces will be listed here without proof.

Theorem 4.10 ([13], Theorem 3): An inner product can be defined on a normed linear space A of dimension three or more if and only if $f(a) = ||f||\ ||a||$ and $g(b) = ||g||\ ||b||$ for linear functional f and g and elements a and b of A imply that there are scalars α and β such that $f(\alpha a+\beta b) + g(\alpha a+\beta b) = ||f+g||\ ||\alpha a+\beta b||$ and $\alpha a+\beta b \neq 0$.

Theorem 4.11 ([13], Theorem 4): An inner product can be defined in a normed linear space of three or more dimensions *if and only if each maximal linear subspace is orthogonal to at least one non-zero element.*

Theorem 4.12 ([13], Theorem 5): *An inner product can be defined in a normed linear space A of three or more dimensions if and only if for any $a \in A$ there is a maximal linear subspace M with $M \perp a$.*

5. Closest Approximations

Let A be a complex normed linear space and B a closed linear subspace. We are interested in the existence, for any $a \in A$, of an element of B which is closest to a. That is, given $a \in A$, can we find $b \in B$ such that $\|a-b\| = \inf_{c \in B} \|a-c\|$? R. A. Hirschfield ([10], Theorem 1) showed that reflexivity of A guarantees the existence of such a b and ([10], Theorem 3) strict convexity (or equivalent strictness of the norm) guarantees uniqueness.

Definition: Let A be a complex normed linear space and B a closed subspace of A. Define (if possible) $P_B : A \rightarrow B$ by $\|a-P_B(a)\| = \inf_{b \in B} \|a-b\|$.

We shall not be immediately interested in whether or not P_B is well defined. P_B shall be called, in the terminology of Hirschfield, the approximation operator.

If A is a Hilbert space, then P_B is well defined and is in fact the orthogonal projection of A onto B. We shall show that if the approximation operator enjoys certain properties possessed by the orthogonal projection, Then A is indeed an inner product space.

Theorem 4.13 ([11], Theorem 1): Let A be a complex normed linear space of dimension greater than two. Suppose the approximation operator satisfies, for a suitable choice of the value of $P_B(a)$,

$$||P_B(a)|| \leqq ||a|| \qquad \text{for all } a\epsilon A$$

for every two dimensional subspace B of A. Then A is an inner product space. (If A is assumed closed, the converse is trivially true.)

Proof: Because of Corollary 1.4.1 we lose no generality by assuming A to be three dimensional. Let B be any 2 dimensional subspace of A and let a be a vector in A but not in B. Then $A = \{b+\alpha a \mid b\epsilon B, \alpha \text{ complex}\}$. Since the hypothesized "suitable choice" of $P_B(a)$ belongs to B we also see that $A = \{b+\alpha(a-P_B(a)) \mid b\epsilon B, \alpha \text{ complex}\}$. For any $c\epsilon A$, let $c = b(c)+\alpha(a-P_B(a))$, $b(c)\epsilon B$. Clearly this representation is unique; that is, if c also equals $b'+\alpha'(a-P_B(a))$ then $b'=b$ and $\alpha'=\alpha$.

We now show that at least one of the values of $P_B(c)$ can be chosen to be b. That is $P_B(c) = b(c)$ for all $c\epsilon A$.

Case I: If $c\epsilon B$, then $\alpha=0$ and $b=c=P_B(c)$.

Case II: If $c \not\epsilon B$, then $\alpha \neq 0$ and for arbitrary $d\epsilon B$ we obtain

$$||c-b|| = ||\alpha(a-P_B(a))|| = |\alpha|\ ||a-P_B(a)||$$
$$\leq |\alpha|\ ||a-(P_B(a)-\alpha^{-1}b+\alpha^{-1}d)|| = ||b+\alpha(a-P_B(a))-d|| = ||c-d||$$

where the inequality holds because $P_B(a)-\alpha^{-1}b+\alpha^{-1}d$ belongs to B. Thus b(c) is a possible value of $P_B(c)$. We now have $c = P_B(c) + \alpha(c)\ (a-P_B(a))$ for all $c \in A$.

We shall now derive linearity of P_B from the uniqueness of the above representation. For $c \in A$ and β complex, we have

$$\beta c = \beta P_B(c) + \beta\alpha(c)\ (a-P_B(a)) \qquad \text{and}$$
$$\beta c = P_B(\beta c) + \alpha(\beta c)\ (a-P_B(a)).$$

Uniqueness implies that $\beta P_B(c) = P_B(\beta c)$. Now for any c and d in A we have

$$c+d = P_B(c) + P_B(d) + (\alpha(c)+\alpha(d))\ (a-P_B(a)) \qquad \text{and}$$
$$c+d = P_B(c+d) + \alpha(c+d)\ (a-P_B(a)).$$

Again by uniqueness, $P_B(c+d) = P_B(c) + P_B(d)$.

Thus for a suitable choice of $P_B(a)$, $P_B(c)$ for any $c \in A$ can be chosen in such a way that P_B is linear in A. Since $||P_B|| \leq 1$ follows from the hypothesis and $||P_B(c)|| = ||c|| = 1$ if $c \in B$, and $||c|| = 1$, we have $||P_B|| = 1$. The existence of a projection of norm one onto the arbitrary two-dimensional subspace B guarantees the existence of an inner product in A by Theorem 4.7.

We return now to define two concepts mentioned in the introduction to this section.

<u>Definition:</u> A norm on a normed linear space A is called <u>strict</u> whenever $||a+b|| = ||a|| + ||b||$ and $a \neq 0$ imply $b = \alpha a$, $\alpha \geq 0$, for all $a, b \epsilon A$.

<u>Definition:</u> A normed vector space is called strictly convex whenever $a, b \epsilon A$, $||a|| = ||b|| = 1$, and $a \neq b$ implies $||\alpha a + (1-\alpha)b|| < 1$, $0 < \alpha < 1$.

Hirschfield shows ([10], Lemma 1, page 48) that normality is equivalent to strict convexity.

<u>Theorem 4.14 ([11], Theorem 2):</u> <u>Let A be a real strictly normed (or, equivalently, strictly convex) linear space of dimension greater than two. Suppose that the approximation operator P_B (which is known to be at most single-valued in strictly normed spaces) satisfies $P_B(a+b) = P_B(a) + P_B(b)$ for all $a, b \epsilon A$ for every one-dimensional subspace B in A. Then A is an inner product space.</u> (Again, if A is complete, the converse holds trivially.)

<u>Proof:</u> As in section four of this chapter we shall call $a \epsilon A$ <u>normal</u> to $b \epsilon A$ if $||a|| \leq ||a+\alpha b||$ for all real α. As before we shall assume A to be three-dimensional.

Let c be any element of A, $c \neq 0$. Denote by (c) the real one dimensional subspace generated by c. The assumption of the theorem means that $P_{(c)}(a+b) = P_{(c)}(a) + P_{(c)}(b)$. Since A is finite dimensional and therefore reflexive, $P_{(c)}$ is defined and, by strict normaltiy, single-valued.

We now show that if $P_{(c)}$ is single-valued it is homogeneous. For every $\alpha \neq 0$, $a \in A$ $||\alpha a - \alpha P_{(c)}(a)||$ $= |\alpha| . ||a - P_{(c)}(a)|| = \inf_{b \in B} ||\alpha a - \alpha b|| = \inf_{b \in B} ||\alpha a - b||$. Likewise $||\alpha a - P_{(c)}(\alpha a)|| = \inf_{b \in B} ||\alpha a - b||$. Thus, since $P_{(c)}$ is single valued, $P_{(c)}(\alpha a) = \alpha P_{(c)}(a)$. We now know that $P_{(c)}$ is linear. We wish to show that it is a continuous projection from A to (c).

That $P_{(c)}$ is idempotent is easy, for let $P_{(c)}(a) = b$. Then $P_{(c)}(P_{(c)}(a)) = P_{(c)}(b) = b = P_{(c)}(a)$ since $b \in B$.

That $P_{(c)}$ is continuous follows from $||a - P_{(c)}(a)||$ $\leq ||a-b||$ for all $a \in A$, $b \in (c)$. For $b = 0$ we get $||a - P_{(c)}(a)|| \leq ||a||$. From this $||P_{(c)}(a)||$ $= ||P_{(c)}(a) - a + a|| \leq ||P_{(c)}(a) - a|| + ||a|| \leq 2||a||$ for all $a \in A$. Thus $P_{(c)}$ is a bounded projection. Hence there exist elements $a \neq 0$, with $P_{(c)}(a) = 0$. For these vectors we have $||a|| = ||a - P_{(c)}(a)|| \leq ||a - \alpha c||$, for all real α. That is, a is normal to c.

Conversely let a be any vector normal to c. Then $||a|| \leq ||a - \alpha c||$, α real. This means, since $P_{(c)}$ is single valued that $P_{(c)}(a) = 0$. Thus $P_{(c)}(a) = 0$ if and only if a is normal to c. Consider now two vectors a and b in A which are normal to c. Then $P_{(c)}(a) = P_{(c)}(b) = 0$. Using additivity we get $P_{(c)}(a+b) = 0$. Thus a+b is normal to c. By Theorem 4.9, A is an inner product space.

Theorem 4.15 ([22], Page 78 footnote): Let A be a Banach space. Let B be a closed linear subspace of A and suppose that for each $a \in A$, $a+B$ has a unique element of minimum norm. (We have seen for example that this will be true if A is reflexive and strictly normed.) Let n(B) be the set of all such elements of minimum norm. If n(B) is a linear subspace for all such B, A is a Hilbert space.

The proof of this theorem will be omitted.

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Vita

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